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Solvability of boundary value problems for impulsive fractional differential equations in Banach spaces

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Abstract

Of concern is the existence of solutions for a class of boundary value problems for impulsive fractional differential equations involving the Caputo fractional derivative in a Banach space. Our approach is based upon the techniques of noncompactness measures and the fixed point theory. Two examples are presented to illustrate the results.

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Keywords: boundary value problem; fractional differential equations; impulsive condition; measure of noncompactness

1 Introduction

The fractional differential equations have received increasing attention during recent years and have been studied extensively (see, *e.g.*, [1–8] and references therein). This is mostly due to the fact that fractional calculus provides an efficient and excellent instrument to describe many practical dynamical phenomena which arise in viscoelasticity, electrochemistry, control, porous media, electromagnetic, *etc.*

A strong motivation for studying impulsive fractional differential equations comes from the fact they can be used to model phenomena that cannot be modeled by traditional initial value problems, such as the dynamics of populations subject to abrupt changes (harvesting, diseases, *etc.*) and mechanics, electrical engineering, medicine biology, ecology, and so on. We refer the readers to [9–15] for the general theory and applications of impulsive differential equations. Recently, Ahmad *et al.* [14] applied the measure of noncompactness to a class of impulsive integrodifferential equations in a Banach space. Liu and Ahmad [15] discussed the existence and uniqueness of solutions for initial value problems of nonlinear singular multiterm impulsive Caputo type fractional differential equations on the half line.

Moreover, some researchers (see [16–20] and the references therein) have addressed the theory of boundary value problems for impulsive fractional differential equations. However, to the best of our knowledge, few papers can be found in the literature for the impulsive fractional differential equations with boundary value conditions in abstract spaces.

In this paper, we study the following fractional impulsive differential equations with boundary value conditions in a Banach space X:

$$^cD_t^q u(t) = f\big(t, u(t)\big), \quad t \in J' := J \setminus \{t_1, t_2, \ldots, t_m\}, J = [0,1], \tag{1.1}$$

$$\Delta u(t_k) = I_k\big(u(t_k)\big), \qquad \Delta u'(t_k) = \widetilde{I}_k\big(u(t_k)\big), \quad k = 1, 2, \ldots, m, \tag{1.2}$$

$$u(0) + u(1) = \alpha \int_0^1 q_1\big(u(s)\big)\, ds, \tag{1.3}$$

$$u'(0) + u'(1) = \beta \int_0^1 q_2\big(u(s)\big)\, ds, \tag{1.4}$$

where $1 < q < 2$, $\alpha, \beta \in \mathbf{R}$. The fractional derivative is understood here in the Caputo sense. Here $f : J \times X \to X$, $I_k, \widetilde{I}_k : X \to X$ $(k = 1, 2, \ldots, m)$, $q_1, q_2 : X \to X$ are appropriate functions to be specified later. The impulsive moments $\{t_k\}$ are given such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\Delta u(t_k)$ represents the jump of function u at t_k, which is defined by $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$, $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

Observe that problem (1.1)-(1.4) reduces to an anti-periodic boundary value problem in [21] for $\alpha = 0$ and $\beta = 0$. If $\alpha \neq 0$ or $\beta \neq 0$, then it reduces to an integral boundary problem.

The rest of this paper is organized as follows. In Section 2, we state some basic concepts, notations and preliminary results about fractional calculus and measure of noncompactness. In Section 3, we discuss a new existence result (Theorem 3.1) for solutions of problem (1.1)-(1.4) on X, and obtain the corresponding result (Theorem 3.2) in $\mathbf{R}$ immediately. We shall give two illustrative examples for our results in Section 4.

2 Preliminaries

Throughout this paper, we denote by X a separable Banach space with norm $\|\cdot\|$. For measurable functions $\xi : J \to X$, define the norm

$$\|\xi\|_{L^p} = \left(\int_J \big\|\xi(t)\big\|^p\, dt\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

Let $L^p(J, X)$ be the Banach space of all Lebesgue measurable functions $\xi : J \to X$ with $\|\xi\|_{L^p} < \infty$.

Let $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $PC(J, X) := \{u : J \to X;\ u \in C(J_k, X),\ k = 0, 1, \ldots, m$, and the right limit $u(t_k^+)$ exists for $k = 1, 2, \ldots, m\}$ with the norm $\|u\|_{PC} := \sup_{t\in J} \|u(t)\|$ and $PC^1(J, X) := \{u : J \to X;\ u \in C^1(J_k, X),\ k = 0, 1, \ldots, m$, and the right limits $u(t_k^+)$, $u'(t_k^+)$ exist for $k = 1, 2, \ldots, m\}$ with the norm $\|u\|_1 := \|u\|_{PC^1} = \max\{\sup_{t\in J} \|u(t)\|, \sup_{t\in J} \|u'(t)\|\}$. Obviously, $PC(J, X)$ and $PC^1(J, X)$ are Banach spaces.

Let us recall the following known definitions. For more details see [4].

Let $g \in L^1([a, \infty), \mathbf{R})$, the set of all integrable functions from $[a, \infty)$ to $\mathbf{R}$.

Definition 2.1 ([4]) The fractional integral of order q with the lower limit a for a function $g(t)$ is defined as

$$_aI_t^q g(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} g(s)\, ds, \quad t > a, q > 0$$

provided the right side is point-wise defined on $[a, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([4]) Riemann-Liouville derivative of order q with the lower limit a for a function $g(t)$ can be written as

$$ {}_a^L D_t^q g(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-q-1} g(s)\, ds, \quad t > a, n-1 < q < n. $$

Definition 2.3 ([4]) The Caputo derivative of order q for the function $g(t)$ can be written as

$$ {}_a^c D_t^q g(t) = {}_a^L D_t^q \left(g(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} g^{(k)}(a) \right), \quad t > a, n-1 < q < n. $$

Remark 2.4

(1) If $g(t) \in C^n[a,\infty)$, then

$$ \begin{aligned} {}_a^c D_t^q g(t) &= \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} g^{(n)}(s)\, ds \\ &= {}_a I_t^{n-q} g^{(n)}(t), \quad t > a, n-1 < q < n. \end{aligned} $$

(2) The Caputo derivative of a constant is equal to zero.

Moreover, from the definition of Caputo derivative, we can obtain the following auxiliary results.

Lemma 2.5 *For $q > 0$, the general solution of fractional differential equation ${}_a^c D_t^q u(t) = 0$ is given by*

$$ u(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}, $$

where $c_i \in \mathbf{R}$, $i = 0,1,2,\ldots,n-1$ ($n = [q]+1$) and $[q]$ denotes the integer part of the real number q.

In view of Lemma 2.5, it follows that the result holds.

Lemma 2.6 *Let $q > 0$, then*

$$ {}_a I_t^q \left({}_a^c D_t^q u\right)(t) = u(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}, $$

where $c_i \in \mathbf{R}$, $i = 0,1,2,\ldots,n-1$, $n = [q]+1$.

Definition 2.7 A function $u \in PC^1(J,X)$ is said to be a solution of (1.1)-(1.4) if u satisfies the equation ${}^c D_t^q u(t) = f(t,u(t))$ on J', and conditions $\Delta u(t_k) = I_k(u(t_k))$, $\Delta u'(t_k) = \widetilde{I}_k(u(t_k))$, $k = 1,2,\ldots,m$ and $u(0) + u(1) = \alpha \int_0^1 q_1(u(s))\, ds$, $u'(0) + u'(1) = \beta \int_0^1 q_2(u(s))\, ds$.

Lemma 2.8 *Let $h : J \to X$ be continuous. A function $u : J \to X$ is a solution of the following fractional integral equation*:

$$u(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\,ds - \frac{1}{2}\sum_{i=1}^m(\frac{1}{\Gamma(q)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}h(s)\,ds + I_i(u(t_i))) \\ \quad - \frac{1}{4}\sum_{i=1}^m(1+2(t-t_i))(\frac{1}{\Gamma(q-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}h(s)\,ds + \widetilde{I}_i(u(t_i))) \\ \quad - \frac{1}{2\Gamma(q)}\int_{t_m}^1(1-s)^{q-1}h(s)\,ds + \frac{1-2t}{4\Gamma(q-1)}\int_{t_m}^1(1-s)^{q-2}h(s)\,ds \\ \quad + \frac{\alpha}{2}\int_0^1 q_1(u(s))\,ds - \frac{\beta(1-2t)}{4}\int_0^1 q_2(u(s))\,ds, \quad t \in J_0, \\ \frac{1}{\Gamma(q)}\int_{t_k}^t (t-s)^{q-1}h(s)\,ds + \frac{1}{2}\sum_{i=1}^k(\frac{1}{\Gamma(q)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}h(s)\,ds + I_i(u(t_i))) \\ \quad - \frac{1}{2}\sum_{i=k+1}^m(\frac{1}{\Gamma(q)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}h(s)\,ds + I_i(u(t_i))) \\ \quad + \frac{1}{2}\sum_{i=1}^k(\frac{1}{\Gamma(q-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}h(s)\,ds + \widetilde{I}_i(u(t_i)))\cdot(t-t_i) \\ \quad - \frac{1}{2}\sum_{i=k+1}^m(\frac{1}{\Gamma(q-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}h(s)\,ds + \widetilde{I}_i(u(t_i)))\cdot(t-t_i) \\ \quad - \frac{1}{4}\sum_{i=1}^m(\frac{1}{\Gamma(q-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}h(s)\,ds + \widetilde{I}_i(u(t_i))) \\ \quad - \frac{1}{2\Gamma(q)}\int_{t_m}^1(1-s)^{q-1}h(s)\,ds + \frac{1-2t}{4\Gamma(q-1)}\int_{t_m}^1(1-s)^{q-2}h(s)\,ds \\ \quad + \frac{\alpha}{2}\int_0^1 q_1(u(s))\,ds - \frac{\beta(1-2t)}{4}\int_0^1 q_2(u(s))\,ds, \quad t \in J_k, k = 1,2,\ldots,m, \end{cases} \tag{2.1}$$

if and only if $u(t)$ is a solution of the problem

$$^cD_t^q u(t) = h(t), \quad t \in J' := J \setminus \{t_1, t_2, \ldots, t_m\}, J = [0,1], \tag{2.2}$$

$$\Delta u(t_k) = I_k\bigl(u(t_k)\bigr), \qquad \Delta u'(t_k) = \widetilde{I}_k\bigl(u(t_k)\bigr), \quad k = 1,2,\ldots,m, \tag{2.3}$$

$$u(0) + u(1) = \alpha\int_0^1 q_1\bigl(u(s)\bigr)\,ds, \tag{2.4}$$

$$u'(0) + u'(1) = \beta\int_0^1 q_2\bigl(u(s)\bigr)\,ds. \tag{2.5}$$

Proof Let u be the solution of (2.2)-(2.5). If $t \in J_0$, then Lemma 2.6 implies that

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\,ds + a_0 + b_0 t, \\ u'(t) &= \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}h(s)\,ds + b_0, \end{aligned} \tag{2.6}$$

for some $a_0, b_0 \in \mathbf{R}$. Obviously, $u(0) = a_0$, $u'(0) = b_0$.

If $t \in J_1$, then Lemma 2.6 implies that

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}h(s)\,ds + c_0 + c_1(t-t_1), \\ u'(t) &= \frac{1}{\Gamma(q-1)}\int_{t_1}^t (t-s)^{q-2}h(s)\,ds + c_1, \end{aligned}$$

for some $c_0, c_1 \in \mathbf{R}$. Thus, we have

$$\begin{aligned} u\bigl(t_1^-\bigr) &= \frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1}h(s)\,ds + a_0 + b_0 t_1, \\ u\bigl(t_1^+\bigr) &= c_0, \end{aligned}$$

$$\begin{aligned}
u'(t_1^-) &= \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} h(s)\,ds + b_0, \\
u'(t_1^+) &= c_1.
\end{aligned}$$

Applying the impulsive condition (2.3), we derive

$$\begin{aligned}
c_0 &= \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} h(s)\,ds + a_0 + b_0 t_1 + I_1\bigl(u(t_1)\bigr), \\
c_1 &= \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} h(s)\,ds + b_0 + \widetilde{I}_1\bigl(u(t_1)\bigr).
\end{aligned}$$

Hence, for $t \in J_1$,

$$\begin{aligned}
u(t) = {} & \frac{1}{\Gamma(q)} \int_{t_1}^{t} (t - s)^{q-1} h(s)\,ds + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} h(s)\,ds \\
& + \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} h(s)\,ds \cdot (t - t_1) + I_1\bigl(u(t_1)\bigr) \\
& + \widetilde{I}_1\bigl(u(t_1)\bigr)(t - t_1) + a_0 + b_0 t.
\end{aligned}$$

If $t \in J_2$, then Lemma 2.6 implies that

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(q)} \int_{t_2}^{t} (t - s)^{q-1} h(s)\,ds + d_0 + d_1(t - t_2), \\
u'(t) &= \frac{1}{\Gamma(q-1)} \int_{t_2}^{t} (t - s)^{q-2} h(s)\,ds + d_1,
\end{aligned}$$

for some $d_0, d_1 \in \mathbf{R}$. Thus, we have

$$\begin{aligned}
u(t_2^-) = {} & \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} h(s)\,ds + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} h(s)\,ds \\
& + \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} h(s)\,ds \cdot (t_2 - t_1) + I_1\bigl(u(t_1)\bigr) \\
& + \widetilde{I}_1\bigl(u(t_1)\bigr)(t_2 - t_1) + a_0 + b_0 t_2, \\
u(t_2^+) = {} & d_0, \\
u'(t_2^-) = {} & \frac{1}{\Gamma(q-1)} \int_{t_1}^{t_2} (t_2 - s)^{q-2} h(s)\,ds + \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} h(s)\,ds \\
& + b_0 + \widetilde{I}_1\bigl(u(t_1)\bigr), \\
u'(t_2^+) = {} & d_1.
\end{aligned}$$

Applying the impulsive condition (2.3), we obtain

$$\begin{aligned}
d_0 = {} & \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} h(s)\,ds + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} h(s)\,ds \\
& + \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} h(s)\,ds \cdot (t_2 - t_1) + I_1\bigl(u(t_1)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ \widetilde{I}_1\big(u(t_1)\big)(t_2 - t_1) + I_2\big(u(t_2)\big) + a_0 + b_0 t_2, \\
d_1 &= \frac{1}{\Gamma(q-1)} \int_{t_1}^{t_2} (t_2 - s)^{q-2} h(s)\,ds + \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} h(s)\,ds \\
&+ b_0 + \widetilde{I}_1\big(u(t_1)\big) + \widetilde{I}_2\big(u(t_2)\big).
\end{aligned}
$$

Hence, for $t \in J_2$,

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(q)} \int_{t_2}^{t} (t - s)^{q-1} h(s)\,ds + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} h(s)\,ds \\
&+ \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} h(s)\,ds + \frac{1}{\Gamma(q-1)} \int_0^{t_1} (t_1 - s)^{q-2} h(s)\,ds \cdot (t - t_1) \\
&+ \frac{1}{\Gamma(q-1)} \int_{t_1}^{t_2} (t_2 - s)^{q-2} h(s)\,ds \cdot (t - t_2) + I_1\big(u(t_1)\big) + I_2\big(u(t_2)\big) \\
&+ \widetilde{I}_1\big(u(t_1)\big)(t - t_1) + \widetilde{I}_2\big(u(t_2)\big)(t - t_2) + a_0 + b_0 t.
\end{aligned}
$$

By repeating the process, for $t \in J_k$, we have

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} h(s)\,ds \\
&+ \sum_{i=1}^{k} \left(\frac{1}{\Gamma(q)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} h(s)\,ds + I_i\big(u(t_i)\big) \right) \\
&+ \sum_{i=1}^{k} \left(\frac{1}{\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds + \widetilde{I}_i\big(u(t_i)\big) \right) \cdot (t - t_i) \\
&+ a_0 + b_0 t, \quad t \in J_k.
\end{aligned}
\tag{2.7}
$$

Now, applying the boundary conditions (2.4), (2.5) to (2.2), we get

$$
\begin{aligned}
b_0 &= -\frac{1}{2} \sum_{i=1}^{m} \left(\frac{1}{\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds + \widetilde{I}_i\big(u(t_i)\big) \right) \\
&+ \frac{\beta}{2} \int_0^1 q_2\big(u(s)\big)\,ds - \frac{1}{2\Gamma(q-1)} \int_{t_m}^{1} (1 - s)^{q-2} h(s)\,ds,
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
a_0 &= -\frac{1}{2\Gamma(q)} \int_{t_m}^{1} (1 - s)^{q-1} h(s)\,ds + \frac{1}{4\Gamma(q-1)} \int_{t_m}^{1} (1 - s)^{q-2} h(s)\,ds \\
&- \frac{1}{2} \sum_{i=1}^{m} \left(\frac{1}{\Gamma(q)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} h(s)\,ds + I_i\big(u(t_i)\big) \right) \\
&- \frac{1}{2} \sum_{i=1}^{m} \left(\frac{1}{2} - t_i \right) \left(\frac{1}{\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds + \widetilde{I}_i\big(u(t_i)\big) \right) \\
&+ \frac{\alpha}{2} \int_0^1 q_1\big(u(s)\big)\,ds - \frac{\beta}{4} \int_0^1 q_2\big(u(s)\big)\,ds.
\end{aligned}
\tag{2.9}
$$

Now, it is clear that (2.6)-(2.9) imply (2.1).

Conversely, if we assume that u satisfies (2.1), by a direct computation, it follows that the solution given by (2.1) satisfies (2.2)-(2.5). This completes the proof. □

Next, we recall that the Hausdorff measure of noncompactness $\chi(\cdot)$ on each bounded subset Ω of Banach space Y is defined by

$$\chi(\Omega) = \inf\{\varepsilon > 0; \Omega \text{ has a finite } \varepsilon\text{-net in } Y\}.$$

This measure of noncompactness satisfies some basic properties as follows.

Lemma 2.9 ([22]) *Let Y be a Banach space and let $U, V \subseteq Y$ be bounded. Then*

(1) $\chi(U) = 0$ *if and only if U is precompact*;
(2) $\chi(U) = \chi(\overline{U}) = \chi(\operatorname{conv} U)$, *where $\overline{U}$ and* conv U *mean the closure and convex hull of U, respectively*;
(3) $\chi(U) \leq \chi(V)$ *if $U \subseteq V$*;
(4) $\chi(U \cup V) \leq \max\{\chi(U), \chi(V)\}$;
(5) $\chi(U + V) \leq \chi(U) + \chi(V)$, *where $U + V = \{x + y; x \in U, y \in V\}$*;
(6) $\chi(\lambda U) = |\lambda|\chi(U)$, *for any $\lambda \in \mathbf{R}$*;
(7) *if the map $Q : D(Q) \subseteq Y \to Z$ is Lipschitz continuous with constant k, then $\chi(Q\Omega) \leq k\chi(\Omega)$ for any bounded subset $\Omega \subseteq D(Q)$, where Z is a Banach space.*

Definition 2.10 A continuous map $Q : W \subseteq Y \to Y$ is said to be a χ-contraction if there exists a positive constant $\nu < 1$ such that $\chi(QU) \leq \nu \cdot \chi(U)$ for any bounded closed subset $U \subseteq W$.

Theorem 2.11 (Darbo-Sadovskii [22]) *If $U \subseteq Y$ is bounded closed and convex, the continuous map $\mathcal{F} : U \to U$ is a χ-contraction, then the map $\mathcal{F}$ has at least one fixed point in U.*

In Section 3, we will use the above fixed point theorem to obtain main result. To this end, we present the following assertion about χ-estimates for a multivalued integral (Theorem 4.2.3 of [23]).

Let 2^Y be the family of all nonempty subset of Y, $G : [0, d] \to 2^Y$ be a multifunction. It is called:

(i) integrable, if it admits a Bochner integrable selection $\varrho : [0, d] \to Y$, $\varrho(t) \in G(t)$ for a.e. $t \in [0, d]$;
(ii) integrably bounded, if there exists a function $\vartheta \in L^1([0, d], Y)$ such that

$$\|G(t)\| := \sup\{\|\varrho\|; \varrho \in G(t)\} \leq \vartheta(t) \quad \text{a.e. } t \in [0, d].$$

Proposition 2.12 *For an integrable, integrably bounded multifunction $G : [0, d] \to 2^X$ where X is a separable Banach space, let*

$$\chi\bigl(G(t)\bigr) \leq m(t), \quad \textit{for a.e. } t \in [0, d],$$

where $m \in L^1_+([0, d])$. Then $\chi(\int_0^t G(s)\,ds) \leq \int_0^t m(s)\,ds$ for all $t \in [0, d]$.

To end this section, we introduce the following PC^1-type Arzela-Ascoli theorem ([24], Theorem 2.1) which will be used in Section 3.

Theorem 2.13 *Let Y be a Banach space and $\mathcal{W} \subset PC^1(J, Y)$. If the following conditions are satisfied*:

(i) *$\mathcal{W}$ is a uniformly bounded subset of $PC^1(J, Y)$*;

(ii) *$\mathcal{W}$ is equicontinuous in (t_k, t_{k+1}), $k = 0, 1, 2, \ldots, m$, where $t_0 = 0$, $t_{m+1} = 1$*;

(iii) *$\mathcal{W}(t) = \{u(t); u \in \mathcal{W}, t \in J \setminus \{t_1, \ldots, t_m\}\}$, $\mathcal{W}(t_k^+) = \{u(t_k^+); u \in \mathcal{W}\}$ and $\mathcal{W}(t_k^-) = \{u(t_k^-); u \in \mathcal{W}\}$ are relatively compact subsets of Y,*

then $\mathcal{W}$ is a relatively compact subset of $PC^1(J, Y)$.

3 Main results

In this section, we will discuss the existence of solutions to (1.1)-(1.4). For this end, we consider the following assumptions.

(H1) (i) $f : J \times X \to X$ satisfies $f(\cdot, v) : J \to X$ is measurable for all $v \in X$ and $f(t, \cdot) : X \to X$ is continuous for a.e. $t \in J$, and there exists a function $\mu \in L^{\frac{1}{\sigma_1}}(J, \mathbf{R}^+)$ ($\sigma_1 \in (0, \frac{q-1}{2})$) such that

$$\|f(t, v)\| \leq \mu(t).$$

(ii) For any bounded set $D \subset X$, there exists a function $\eta \in L^{\frac{1}{\sigma_2}}(J, \mathbf{R}^+)$ ($\sigma_2 \in (0, q-1)$) such that

$$\chi\bigl(f(t, D)\bigr) \leq \eta(t)\chi(D).$$

(H2) The functions $I_k, \widetilde{I}_k : X \to X$ ($k = 1, 2, \ldots, m$) are continuous and there exist positive constants $l_k, \widetilde{l}_k$ such that

$$\begin{aligned}
\bigl\|I_k(u) - I_k(v)\bigr\| &\leq l_k\|u - v\|, \quad u, v \in X, \\
\bigl\|\widetilde{I}_k(u) - \widetilde{I}_k(v)\bigr\| &\leq \widetilde{l}_k\|u - v\|, \quad u, v \in X.
\end{aligned}$$

(H3) There exist constants $M_{q_1}, M_{q_2} \in \mathbf{R}^+$ such that

$$\bigl\|q_i(u_1) - q_i(u_2)\bigr\| \leq M_{q_i}\|u_1 - u_2\|, \quad u_1, u_2 \in X, i = 1, 2.$$

Theorem 3.1 *Assume that* (H1)-(H3) *are satisfied, then there exists a solution of* (1.1)-(1.4) *on J provided that*

$$\begin{aligned}
\kappa := {}& \frac{(3m+1)q - m + 5}{2\Gamma(q)} \left(\frac{1 - \sigma_2}{q - \sigma_2 - 1}\right)^{1-\sigma_2} \|\eta\|_{L^{\frac{1}{\sigma_2}}} \\
&+ \sum_{i=1}^{m} \frac{2l_i + 3\widetilde{l}_i}{4} + \frac{\alpha M_{q_1} + \beta M_{q_2}}{2} < 1.
\end{aligned}$$

Proof Consider the operator $\mathcal{F} : PC^1(J,X) \to PC^1(J,X)$ defined by

$$(\mathcal{F}u)(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))\,ds \\ \quad - \frac{1}{2}\sum_{i=1}^m (\frac{1}{\Gamma(q)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f(s,u(s))\,ds + I_i(u(t_i))) \\ \quad - \frac{1}{4}\sum_{i=1}^m (1+2(t-t_i))(\frac{1}{\Gamma(q-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f(s,u(s))\,ds + \widetilde{I}_i(u(t_i))) \\ \quad - \frac{1}{2\Gamma(q)}\int_{t_m}^1 (1-s)^{q-1}f(s,u(s))\,ds + \frac{1-2t}{4\Gamma(q-1)}\int_{t_m}^1 (1-s)^{q-2}f(s,u(s))\,ds \\ \quad + \frac{\alpha}{2}\int_0^1 q_1(u(s))\,ds - \frac{\beta(1-2t)}{4}\int_0^1 q_2(u(s))\,ds, \quad t \in J_0, \\ \frac{1}{\Gamma(q)}\int_{t_k}^t (t-s)^{q-1}f(s,u(s))\,ds \\ \quad + \frac{1}{2}\sum_{i=1}^k (\frac{1}{\Gamma(q)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f(s,u(s))\,ds + I_i(u(t_i))) \\ \quad - \frac{1}{2}\sum_{i=k+1}^m (\frac{1}{\Gamma(q)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f(s,u(s))\,ds + I_i(u(t_i))) \\ \quad + \frac{1}{2}\sum_{i=1}^k (\frac{1}{\Gamma(q-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f(s,u(s))\,ds + \widetilde{I}_i(u(t_i))) \cdot (t-t_i) \\ \quad - \frac{1}{2}\sum_{i=k+1}^m (\frac{1}{\Gamma(q-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f(s,u(s))\,ds + \widetilde{I}_i(u(t_i))) \cdot (t-t_i) \\ \quad - \frac{1}{4}\sum_{i=1}^m (\frac{1}{\Gamma(q-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f(s,u(s))\,ds + \widetilde{I}_i(u(t_i))) \\ \quad - \frac{1}{2\Gamma(q)}\int_{t_m}^1 (1-s)^{q-1}f(s,u(s))\,ds + \frac{1-2t}{4\Gamma(q-1)}\int_{t_m}^1 (1-s)^{q-2}f(s,u(s))\,ds \\ \quad + \frac{\alpha}{2}\int_0^1 q_1(u(s))\,ds - \frac{\beta(1-2t)}{4}\int_0^1 q_2(u(s))\,ds, \quad t \in J_k, k = 1,2,\ldots,m. \end{cases} \tag{3.1}$$

Clearly, $\mathcal{F}$ is well defined and the fixed point of $\mathcal{F}$ is the solution of problem (1.1)-(1.4) by Lemma 2.8.

The operator $\mathcal{F}$ can be rewritten in the form $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, for $t \in J_k$ ($k = 0,1,2,\ldots,m$), where

$$\begin{aligned} (\mathcal{F}_1u)(t) &= \frac{1}{\Gamma(q)}\int_{t_k}^t (t-s)^{q-1}f\bigl(s,u(s)\bigr)\,ds \\ &\quad + \frac{1}{2\Gamma(q)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f\bigl(s,u(s)\bigr)\,ds \\ &\quad - \frac{1}{2\Gamma(q)}\sum_{i=k+1}^m \int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f\bigl(s,u(s)\bigr)\,ds \\ &\quad + \frac{1}{2\Gamma(q-1)}\sum_{i=1}^k \biggl(\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f\bigl(s,u(s)\bigr)\,ds\biggr) \cdot (t-t_i) \\ &\quad - \frac{1}{2\Gamma(q-1)}\sum_{i=k+1}^m \biggl(\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f\bigl(s,u(s)\bigr)\,ds\biggr) \cdot (t-t_i) \\ &\quad - \frac{1}{4\Gamma(q-1)}\sum_{i=1}^m \int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f\bigl(s,u(s)\bigr)\,ds \\ &\quad - \frac{1}{2\Gamma(q)}\int_{t_m}^1 (1-s)^{q-1}f\bigl(s,u(s)\bigr)\,ds + \frac{1-2t}{4\Gamma(q-1)}\int_{t_m}^1 (1-s)^{q-2}f\bigl(s,u(s)\bigr)\,ds, \\ (\mathcal{F}_2u)(t) &= \frac{1}{2}\sum_{i=1}^k I_i\bigl(u(t_i)\bigr) - \frac{1}{2}\sum_{i=k+1}^m I_i\bigl(u(t_i)\bigr) - \frac{1}{4}\sum_{i=1}^m \widetilde{I}_i\bigl(u(t_i)\bigr) \\ &\quad + \frac{1}{2}\sum_{i=1}^k \widetilde{I}_i\bigl(u(t_i)\bigr) \cdot (t-t_i) - \frac{1}{2}\sum_{i=k+1}^m \widetilde{I}_i\bigl(u(t_i)\bigr) \cdot (t-t_i) \\ &\quad + \frac{\alpha}{2}\int_0^1 q_1\bigl(u(s)\bigr)\,ds - \frac{\beta(1-2t)}{4}\int_0^1 q_2\bigl(u(s)\bigr)\,ds. \end{aligned}$$

Then

$$
\begin{aligned}
(\mathcal{F}_1 u)'(t) &= \frac{1}{\Gamma(q-1)} \int_{t_k}^{t} (t-s)^{q-2} f\big(s,u(s)\big)\,ds \\
&\quad + \frac{1}{2\Gamma(q-1)} \sum_{i=1}^{k} \left(\int_{t_{i-1}}^{t_i} (t_i-s)^{q-2} f\big(s,u(s)\big)\,ds \right) \\
&\quad - \frac{1}{2\Gamma(q-1)} \sum_{i=k+1}^{m+1} \left(\int_{t_{i-1}}^{t_i} (t_i-s)^{q-2} f\big(s,u(s)\big)\,ds \right) \\
(\mathcal{F}_2 u)'(t) &= \frac{1}{2} \sum_{i=1}^{k} \widetilde{I}_i\big(u(t_i)\big) - \frac{1}{2} \sum_{i=k+1}^{m} \widetilde{I}_i\big(u(t_i)\big) \\
&\quad + \frac{\beta}{2} \int_0^1 q_2\big(u(s)\big)\,ds.
\end{aligned}
$$

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC^1(J,X)$ as $n \to \infty$. Since f satisfies (H1)(i), for almost every $t \in J$, we get $f(t,u_n(t)) \to f(t,u(t))$ as $n \to \infty$. Moreover, $\|f(t,u_n(t)) - f(t,u(t))\| \le 2\mu(t)$.

It follows from Lebesgue's dominated convergence theorem that

$$
\big\|(\mathcal{F}_1 u_n)(t) - (\mathcal{F}_1 u)(t)\big\|_1 \to 0, \quad \text{as } n \to \infty.
$$

Moreover, noting that (H2) and (H3), we have $\|(\mathcal{F}_2 u_n)(t) - (\mathcal{F}_2 u)(t)\|_1 \to 0$ as $n \to \infty$. Now we can see that $\mathcal{F}$ is continuous.

Let $B_r = \{u \in PC^1(J,X); \|u\|_1 \le r\}$, where $r \ge \frac{C_0}{1-C_1}$ and

$$
\begin{aligned}
C_0 &= \frac{(1-\sigma_1)^{1-\sigma_1} \cdot \|\mu\|_{L^{\frac{1}{\sigma_1}}}}{\Gamma(q)(q-\sigma_1-1)^{1-\sigma_1}} \cdot \frac{(3m+1)q - m + 5}{4} \\
&\quad + \sum_{i=1}^{m} \frac{2\|I_i(0)\| + 3\|\widetilde{I}_i(0)\|}{4} + \frac{\alpha\|q_1(0)\| + \beta\|q_2(0)\|}{2}, \\
C_1 &= \sum_{i=1}^{m} \frac{2l_i + 3\widetilde{l}_i}{4} + \frac{\alpha M_{q_1} + \beta M_{q_2}}{2}.
\end{aligned}
$$

For $u \in B_r$, $i = 1, 2, \ldots, m+1$, using (H1)(i) and the Hölder inequality, we have

$$
\begin{aligned}
\left\| \int_{t_{i-1}}^{t_i} (t_i-s)^{q-1} f\big(s,u(s)\big)\,ds \right\| &\le \left(\int_{t_{i-1}}^{t_i} (t_i-s)^{\frac{q-1}{1-\sigma_1}}\,ds \right)^{1-\sigma_1} \left(\int_{t_{i-1}}^{t_i} \big(\mu(s)\big)^{\frac{1}{\sigma_1}}\,ds \right)^{\sigma_1} \\
&< \left(\frac{1-\sigma_1}{q-\sigma_1} \right)^{1-\sigma_1} \|\mu\|_{L^{\frac{1}{\sigma_1}}}, \\
\left\| \int_{t_{i-1}}^{t_i} (t_i-s)^{q-2} f\big(s,u(s)\big)\,ds \right\| &\le \left(\int_{t_{i-1}}^{t_i} (t_i-s)^{\frac{q-2}{1-\sigma_1}}\,ds \right)^{1-\sigma_1} \left(\int_{t_{i-1}}^{t_i} \big(\mu(s)\big)^{\frac{1}{\sigma_1}}\,ds \right)^{\sigma_1} \\
&< \left(\frac{1-\sigma_1}{q-\sigma_1-1} \right)^{1-\sigma_1} \|\mu\|_{L^{\frac{1}{\sigma_1}}}.
\end{aligned}
$$

Therefore

$$\begin{aligned}
\|(\mathcal{F}_1u)(t)\| &< \frac{(1-\sigma_1)^{1-\sigma_1}\cdot\|\mu\|_{L^{\frac{1}{\sigma_1}}}}{\Gamma(q)(q-\sigma_1-1)^{1-\sigma_1}}\cdot\frac{(3m+1)q-m+5}{4},\\
\|(\mathcal{F}_1u)'(t)\| &< \frac{(1-\sigma_1)^{1-\sigma_1}\cdot\|\mu\|_{L^{\frac{1}{\sigma_1}}}}{\Gamma(q-1)(q-\sigma_1-1)^{1-\sigma_1}}\cdot\frac{m+3}{2},
\end{aligned}$$

then

$$\|(\mathcal{F}_1u)(t)\|_1 \le \frac{(1-\sigma_1)^{1-\sigma_1}\cdot\|\mu\|_{L^{\frac{1}{\sigma_1}}}}{\Gamma(q)(q-\sigma_1-1)^{1-\sigma_1}}\cdot\frac{(3m+1)q-m+5}{4}. \tag{3.2}$$

Moreover, by (H2) and (H3), we obtain

$$\begin{aligned}
\|(\mathcal{F}_2u)(t)\| &< \sum_{i=1}^{m}\frac{(2l_i+3\widetilde{l}_i)}{4}\|u(t_i)\| + \frac{\alpha M_{q_1}+\beta M_{q_2}}{2}\int_0^1\|u(s)\|\,ds\\
&\quad + \sum_{i=1}^{m}\frac{2\|I_i(0)\|+3\|\widetilde{I}_i(0)\|}{4} + \frac{\alpha\|q_1(0)\|+\beta\|q_2(0)\|}{2}
\end{aligned}$$

and

$$\|(\mathcal{F}_2u)'(t)\| \le \frac{1}{2}\sum_{i=1}^{m}\widetilde{l}_i\|u(t_i)\| + \frac{\beta M_{q_2}}{2}\int_0^1\|u(s)\|\,ds + \frac{1}{2}\sum_{i=1}^{m}\|\widetilde{I}_i(0)\| + \frac{\beta\|q_2(0)\|}{2},$$

therefore

$$\begin{aligned}
\|(\mathcal{F}_2u)(t)\|_1 &\le \left(\sum_{i=1}^{m}\frac{2l_i+3\widetilde{l}_i}{4} + \frac{\alpha M_{q_1}+\beta M_{q_2}}{2}\right)r + \sum_{i=1}^{m}\frac{2\|I_i(0)\|+3\|\widetilde{I}_i(0)\|}{4}\\
&\quad + \frac{\alpha\|q_1(0)\|+\beta\|q_2(0)\|}{2}.
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have $\|(\mathcal{F}u)(t)\|_1 \le C_0 + C_1 r \le r$. This shows that $\mathcal{F}B_r \subset B_r$.

Let $u \in B_r$, for any $t_k < \tau_2 < \tau_1 \le t_{k+1}$, we have

$$\begin{aligned}
&\left\|\frac{1}{\Gamma(q)}\int_{t_k}^{\tau_1}(\tau_1-s)^{q-1}f(s,u(s))\,ds - \frac{1}{\Gamma(q)}\int_{t_k}^{\tau_2}(\tau_2-s)^{q-1}f(s,u(s))\,ds\right\|\\
&\quad\le \frac{1}{\Gamma(q)}\int_{t_k}^{\tau_2}\left[(\tau_1-s)^{q-1}-(\tau_2-s)^{q-1}\right]\mu(s)\,ds + \frac{1}{\Gamma(q)}\int_{\tau_2}^{\tau_1}(\tau_1-s)^{q-1}\mu(s)\,ds\\
&\quad\to 0, \quad \text{as } \tau_2 \to \tau_1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(\mathcal{F}u)(\tau_1)-(\mathcal{F}u)(\tau_2)\|\\
&\quad\le \left\|\frac{1}{\Gamma(q)}\int_{t_k}^{\tau_1}(\tau_1-s)^{q-1}f(s,u(s))\,ds - \frac{1}{\Gamma(q)}\int_{t_k}^{\tau_2}(\tau_2-s)^{q-1}f(s,u(s))\,ds\right\|
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(\tau_1-\tau_2)}{2}\left[\sum_{i=1}^{m}\bigl\|\widetilde{I}_i\bigl(u(t_i)\bigr)\bigr\| + \beta\int_0^1 \bigl\|q_2\bigl(u(s)\bigr)\bigr\|\,ds\right] \\
&+ \frac{(\tau_1-\tau_2)}{2\Gamma(q-1)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\mu(s)\,ds \\
&\to 0, \quad \text{as } \tau_2 \to \tau_1.
\end{aligned}
$$

Moreover, from the Hölder inequality, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(q-1)}\left\|\int_{t_k}^{\tau_2}\bigl[(\tau_1-s)^{q-2}-(\tau_2-s)^{q-2}\bigr]\mu(s)\,ds\right\| \\
&\quad\le \frac{1}{\Gamma(q-1)}\left(\int_{t_k}^{\tau_2}\bigl|(\tau_1-s)^{q-2}-(\tau_2-s)^{q-2}\bigr|^{\frac{1}{1-\sigma_1}}\,ds\right)^{1-\sigma_1}\cdot\|\mu\|_{L^{\frac{1}{\sigma_1}}} \\
&\quad= \frac{2-q}{\Gamma(q-1)}\cdot\|\mu\|_{L^{\frac{1}{\sigma_1}}}\left(\int_{t_k}^{\tau_2}\left[\int_{\tau_2}^{\tau_1}(\zeta-s)^{q-3}\,d\zeta\right]^{\frac{1}{1-\sigma_1}}ds\right)^{1-\sigma_1} \\
&\quad\le M\left(\int_{t_k}^{\tau_2}\bigl((\tau_2-s)^{\theta}-(\tau_1-s)^{\theta}\bigr)\,ds\right)^{1-\sigma_1} \\
&\quad\le \frac{M}{(1+\theta)^{1-\sigma_1}}\bigl[(\tau_1-\tau_2)^{1+\theta}-(\tau_1-t_k)^{1+\theta}+(\tau_2-t_k)^{1+\theta}\bigr]^{1-\sigma_1} \\
&\quad\to 0, \quad \text{as } \tau_2\to\tau_1,
\end{aligned}
$$

where $M > 0$ is a constant and $\theta = \frac{q-2-\sigma_1}{1-\sigma_1}$. Similarly,

$$
\begin{aligned}
\int_{\tau_2}^{\tau_1}(\tau_1-s)^{q-2}\mu(s)\,ds &\le \left(\int_{\tau_2}^{\tau_1}(\tau_1-s)^{\frac{q-2}{1-\sigma_1}}\,ds\right)^{1-\sigma_1}\cdot\|\mu\|_{L^{\frac{1}{\sigma_1}}} \\
&= \left(\frac{1-\sigma_1}{q-1-\sigma_1}\right)^{1-\sigma_1}(\tau_1-\tau_2)^{q-1-\sigma_1}\cdot\|\mu\|_{L^{\frac{1}{\sigma_1}}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\bigl\|(\mathcal{F}u)'(\tau_1)-(\mathcal{F}u)'(\tau_2)\bigr\| \\
&\quad\le \frac{1}{\Gamma(q-1)}\left\|\int_{t_k}^{\tau_1}(\tau_1-s)^{q-2}f\bigl(s,u(s)\bigr)\,ds-\int_{t_k}^{\tau_2}(\tau_2-s)^{q-2}f\bigl(s,u(s)\bigr)\,ds\right\| \\
&\quad\le \frac{1}{\Gamma(q-1)}\left\|\int_{t_k}^{\tau_2}\bigl[(\tau_1-s)^{q-2}-(\tau_2-s)^{q-2}\bigr]\mu(s)\,ds\right\| + \frac{1}{\Gamma(q-1)}\int_{\tau_2}^{\tau_1}(\tau_1-s)^{q-2}\mu(s)\,ds \\
&\quad\to 0, \quad \text{as } \tau_2\to\tau_1.
\end{aligned}
$$

This shows that the set $\{(\mathcal{F}u)(\cdot); u \in B_r\}$ is equicontinuous.

If $W \subset PC^1(J,X)$ is bounded and the elements of W' are equicontinuous on each $J_k = (t_k, t_{k+1}]$ ($k = 0,1,\ldots,m$), we can consider the measure of noncompactness χ_{pc^1} on the space $PC^1(J,X)$ in the following way:

$$
\chi_{pc^1}(W) = \max\Bigl\{\max_{0\le k\le m}\chi\bigl(W(J_k)\bigr), \max_{0\le k\le m}\chi\bigl(W'(J_k)\bigr)\Bigr\}.
$$

Let $\Omega \subset B_r$ be a nonempty, bounded set. Clearly, we can see

$$\begin{aligned}
&\chi\left(\int_{t_k}^{t}(t-s)^{q-1}f\bigl(s,\Omega(s)\bigr)\,ds\right)\\
&\quad\leq \int_{t_k}^{t}(t-s)^{q-1}\eta(s)\chi\bigl(\Omega(s)\bigr)\,ds, \quad t\in J_k, k=0,1,2,\ldots,m,\\
&\chi\left(\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f\bigl(s,\Omega(s)\bigr)\,ds\right)\\
&\quad\leq \int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\eta(s)\chi\bigl(\Omega(s)\bigr)\,ds, \quad i=1,2,\ldots,m+1.
\end{aligned}\tag{3.4}$$

Moreover, for $i=1,2,\ldots,m+1$, we set

$$\widetilde{\mathcal{F}}_i(\Omega)=\left\{\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f\bigl(s,u(s)\bigr)\,ds; u\in\Omega\right\}.$$

Consider the multifunction $s\in[t_{i-1},t_i]\multimap H_i(s)$

$$H_i(s)=\bigl\{(t_i-s)^{q-2}f\bigl(s,u(s)\bigr); u\in\Omega\bigr\}, \quad i=1,2,\ldots,m+1.$$

Obviously, H_i is integrable and from (H1)(i) it follows that H_i is integrably bounded. Moreover, noting that (H1)(ii), we have the following estimate for a.e. $s\in[t_{i-1},t_i]$, $i=1,2,\ldots,m+1$

$$\chi\bigl(H_i(s)\bigr)=\chi\bigl(\bigl\{(t_i-s)^{q-2}f\bigl(s,u(s)\bigr); u\in\Omega\bigr\}\bigr)\leq(t_i-s)^{q-2}\eta(s)\chi\bigl(\Omega(s)\bigr).$$

Applying Proposition 2.12, we obtain

$$\chi\bigl(\widetilde{\mathcal{F}}_i(\Omega)\bigr)=\chi\left(\int_{t_{i-1}}^{t_i}H_i(s)\,ds\right)\leq\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\eta(s)\chi\bigl(\Omega(s)\bigr)\,ds.\tag{3.5}$$

Therefore, combining with (3.4), (3.5) and the Hölder inequality, we have

$$\begin{aligned}
\chi\bigl((\mathcal{F}_1\Omega)(t)\bigr)\leq{}&\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\eta(s)\chi\bigl(\Omega(s)\bigr)\,ds\\
&+\frac{1}{2\Gamma(q)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\eta(s)\chi\bigl(\Omega(s)\bigr)\,ds\\
&+\frac{3}{4\Gamma(q-1)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\eta(s)\chi\bigl(\Omega(s)\bigr)\,ds\\
&+\frac{1}{2\Gamma(q)}\int_{t_m}^{1}(1-s)^{q-1}\eta(s)\chi\bigl(\Omega(s)\bigr)\,ds\\
&+\frac{1}{4\Gamma(q-1)}\int_{t_m}^{1}(1-s)^{q-2}\eta(s)\chi\bigl(\Omega(s)\bigr)\,ds\\
<{}&\frac{(3m+1)q-m+5}{2\Gamma(q)}\left(\frac{1-\sigma_2}{q-\sigma_2-1}\right)^{1-\sigma_2}\|\eta\|_{L^{\frac{1}{\sigma_2}}}\cdot\chi_{pc^1}(\Omega).
\end{aligned}\tag{3.6}$$

Similarly,

$$\begin{aligned}\chi\left((\mathcal{F}_1\Omega)'(t)\right) &\le \frac{1}{\Gamma(q-1)}\int_{t_k}^{t}(t-s)^{q-2}\eta(s)\chi\left(\Omega(s)\right)ds \\ &\quad + \frac{1}{2\Gamma(q-1)}\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\eta(s)\chi\left(\Omega(s)\right)ds \\ &\le \frac{(m+3)}{2\Gamma(q-1)}\left(\frac{1-\sigma_2}{q-\sigma_2-1}\right)^{1-\sigma_2}\|\eta\|_{L^{\frac{1}{\sigma_2}}}\cdot\chi_{pc^1}(\Omega).\end{aligned} \tag{3.7}$$

Since $\mathcal{F}_1\Omega$ and $(\mathcal{F}_1\Omega)'$ are equicontinuous on every J_k, by Proposition 7.3 of [25], we find that

$$\begin{aligned}\chi_{pc^1}(\mathcal{F}_1\Omega) &= \max\left\{\max_{0\le k\le m}\chi\left((\mathcal{F}_1\Omega)(J_k)\right), \max_{0\le k\le m}\chi\left((\mathcal{F}_1\Omega)'(J_k)\right)\right\} \\ &= \max\left\{\max_{0\le k\le m}\max_{t\in J_k}\chi\left((\mathcal{F}_1\Omega)(t)\right), \max_{0\le k\le m}\max_{t\in J_k}\chi\left((\mathcal{F}_1\Omega)'(t)\right)\right\}.\end{aligned}$$

Then, according to inequalities (3.6) and (3.7), we have

$$\chi_{pc^1}(\mathcal{F}_1\Omega) < \frac{(3m+1)q-m+5}{2\Gamma(q)}\left(\frac{1-\sigma_2}{q-\sigma_2-1}\right)^{1-\sigma_2}\|\eta\|_{L^{\frac{1}{\sigma_2}}}\cdot\chi_{pc^1}(\Omega). \tag{3.8}$$

Moreover, by (H2) and (H3), for any $u, v \in B_r$,

$$\left\|(\mathcal{F}_2u)(t)-(\mathcal{F}_2v)(t)\right\|_1 \le C_1\|u-v\|_1, \quad t \in J.$$

This means that $\mathcal{F}_2$ is Lipschitz continuous with Lipschitz constant C_1. It follows from (7) in Lemma 2.9 that

$$\chi_{pc^1}(\mathcal{F}_2\Omega) \le C_1\chi_{pc^1}(\Omega). \tag{3.9}$$

Therefore, from (3.8) and (3.9), we get

$$\chi_{pc^1}(\mathcal{F}\Omega) \le \chi_{pc^1}(\mathcal{F}_1\Omega) + \chi_{pc^1}(\mathcal{F}_2\Omega) \le \kappa\chi_{pc^1}(\Omega).$$

Hence, $\mathcal{F}$ is a χ-contraction on B_r by Definition 2.10. According to Theorem 2.11, the operator $\mathcal{F}$ has at least one fixed point u in B_r, which is a solution of problem (1.1)-(1.4). This ends the proof. □

When $X = \mathbf{R}$, we rely on Schauder's fixed point theorem, which gives an existence result for solutions of problem (1.1)-(1.4) under the following assumptions.

(H1′) $f : J \times \mathbf{R} \to \mathbf{R}$ is jointly continuous, and there exists a function $\mu \in L^{\frac{1}{\sigma_1}}(J, \mathbf{R}^+)$ ($\sigma_1 \in (0, \frac{q-1}{2})$) such that $|f(t,v)| \le \mu(t)$.

(H2′) The functions $I_k, \widetilde{I}_k : \mathbf{R} \to \mathbf{R}$ are continuous and there exist positive constants l_k, $\widetilde{l}_k$ such that $|I_k(u)-I_k(v)| \le l_k|u-v|$ and $|\widetilde{I}_k(u)-\widetilde{I}_k(v)| \le \widetilde{l}_k|u-v|$, $u, v \in \mathbf{R}$, $k = 1, 2, \ldots, m$.

(H3′) There exist constants $M_{q_1}, M_{q_2} \in \mathbf{R}^+$ such that $|q_i(u_1)-q_i(u_2)| \le M_{q_i}|u_1-u_2|$, $u_1, u_2 \in \mathbf{R}$, $i = 1, 2$.

Theorem 3.2 *Assume that the assumptions* (H1′)-(H3′) *hold. Then problem* (1.1)-(1.4) *has at least one solution on J provided that*

$$\widetilde{\kappa} := \sum_{i=1}^{m} \frac{2l_i + 3\widetilde{l}_i}{4} + \frac{\alpha M_{q_1} + \beta M_{q_2}}{2} < 1.$$

Proof Let $\mathcal{F} : PC^1(J,\mathbf{R}) \to PC^1(J,\mathbf{R})$ be defined as in the proof of Theorem 3.1. In the proof of Theorem 3.1, we can see that $\mathcal{F}$ is continuous, $\mathcal{F}$ maps bounded sets B_r into bounded sets and equicontinuous sets. Then we can deduce that $\mathcal{F}$ is compact on B_r as a result of the PC^1-type Arzela-Ascoli theorem (see Theorem 2.13 in the case of $X = \mathbf{R}$).

As a consequence of Schauder's fixed point theorem, we conclude that $\mathcal{F}$ has a fixed point, that is, problem (1.1)-(1.4) has at least one solution on $\mathbf{R}$. The proof is complete. □

4 Applications

In this section, we give two examples to illustrate the usefulness of our main results.

Example 4.1 Let $X = L^2([0,\pi])$. Consider the following impulsive integral boundary problem:

$$\begin{cases} \partial_t^q v(t,x) = a(t)\cos(|v(t,x)|), \quad t \in J' = [0,1] \setminus \{t_1, t_2, \ldots, t_m\}, \\ \triangle v(t_k,x) = \int_0^{\pi} \rho_k(x,y)\,dy \cdot \sin^2(v(t_k,x)), \quad k = 1,2,\ldots,m, \\ \triangle v'(t_k,x) = \frac{1}{2^k} \cdot \frac{|v(t_k,x)|}{1+|v(t_k,x)|}, \quad k = 1,2,\ldots,m, \\ v(0,x) + v(1,x) = \int_0^1 b(s)e^{-|v(s,x)|}\,ds, \\ v'(0,x) + v'(1,x) = \int_0^1 c(s) \cdot \frac{|v(s,x)|}{1+|v(s,x)|}\,ds, \end{cases} \tag{4.1}$$

where $q \in (1,2)$, $x \in [0,\pi]$, $a(t) \in L^{\frac{1}{\sigma}}(J,\mathbf{R}^+)$, $\sigma \in (0, \frac{q-1}{2})$, $b(t)$, $c(t)$ are bounded functions on $\mathbf{R}$ such that $|b(t)| \le b^*$, $|c(t)| \le c^*$, the functions $\rho_k(x,y)$, $y \in [0,\pi]$ ($k = 1,2,\ldots,m$) are measurable and there exists a constant $\overline{N}$ such that $\int_0^{\pi} \|\rho_k(x,y)\|\,dy \le \overline{N}$.

For $t \in [0,1]$, $x \in [0,\pi]$, $k = 1,2,\ldots,m$, we set

$$\begin{aligned} &u(t)(x) = v(t,x), \\ &f\bigl(t,u(t)\bigr)(x) = a(t)\cos\bigl(\bigl|u(t)(x)\bigr|\bigr), \\ &I_k\bigl(u(t_k)\bigr)(x) = \int_0^{\pi} \rho_k(x,y)\,dy \cdot \sin^2\bigl(u(t_k)(x)\bigr), \quad k = 1,2,\ldots,m, \\ &\widetilde{I}_k\bigl(u(t_k)\bigr)(x) = \frac{1}{2^k} \cdot \frac{|u(t_k)(x)|}{1+|u(t_k)(x)|}, \quad k = 1,2,\ldots,m, \\ &q_1\bigl(u(t)(x)\bigr) = b(t)e^{-|u(t)(x)|}, \\ &q_2\bigl(u(t)(x)\bigr) = c(t) \cdot \frac{|u(t)(x)|}{1+|u(t)(x)|}. \end{aligned}$$

Then we can rewrite (4.1) in the abstract form (1.1)-(1.4).

Obviously, we have $\|f(t,u(t))(x)\| \le a(t)$ for $t \in [0,1]$. For any $u_1, u_2 \in X$,

$$\bigl\|f\bigl(t,u_1(t)\bigr) - f\bigl(t,u_2(t)\bigr)\bigr\| \le a(t)\bigl\|u_1(t) - u_2(t)\bigr\|,$$

which implies that for any bounded set $D \subset X$, $\chi(f(t,D)) \le a(t)\chi(D)$, $t \in [0,1]$.

Moreover, for any $u, \widetilde{u} \in X$,

$$\begin{aligned}
&\big\| I_k\big(u(t_k)\big) - I_k\big(\widetilde{u}(t_k)\big)\big\| \le 2\overline{N}\big\|u(t_k) - \widetilde{u}(t_k)\big\|, \\
&\big\| \widetilde{I}_k\big(u(t_k)\big) - \widetilde{I}_k\big(\widetilde{u}(t_k)\big)\big\| \le \frac{1}{2^k}\big\|u(t_k) - \widetilde{u}(t_k)\big\|, \\
&\big\| q_1\big(u(t)\big) - q_1\big(\widetilde{u}(t)\big)\big\| \le b^*\big\|u(t) - \widetilde{u}(t)\big\|, \\
&\big\| q_2\big(u(t)\big) - q_2\big(\widetilde{u}(t)\big)\big\| \le c^*\big\|u(t) - \widetilde{u}(t)\big\|.
\end{aligned}$$

Suppose further that

$$\frac{(3m+1)q - m + 5}{2\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma-1}\right)^{1-\sigma}\|a\|_{L^{\frac{1}{\sigma}}} + m\overline{N} + \frac{3}{4}\left(1 - \frac{1}{2^m}\right) + \frac{b^* + c^*}{2} < 1,$$

then problem (4.1) has at least a solution by Theorem 3.1.

Example 4.2 Let $X = \mathbf{R}$. Consider the following impulsive anti-periodic boundary problem of fractional order:

$$\begin{cases}
{}^cD_t^{\frac{3}{2}} u(t) = e^t \arctan(u(t)) + t\int_0^t e^{-3|u(s)|}\,ds, \quad t \in [0,1] \setminus \{\frac{1}{3}\}, \\
\triangle u(\frac{1}{3}) = \frac{e^{-|u(\frac{1}{3})|}}{10+e^t}, \\
\triangle u'(\frac{1}{3}) = \frac{2+|u(\frac{1}{3})|}{7+|u(\frac{1}{3})|}, \\
u(0,x) + u(1,x) = 0, \\
u'(0,x) + u'(1,x) = 0.
\end{cases} \tag{4.2}$$

Set $q = \frac{3}{2}$, $\sigma = \frac{1}{6}$ and $f(t,u(t)) = e^t \arctan(u(t)) + t\int_0^t e^{-3|u(s)|}\,ds$. Obviously, $\|f(t,u(t))\| < \frac{\pi}{2}(e^t + t^2) \in L^6([0,1],\mathbf{R})$. Moreover, $l_1 = \frac{1}{10}$, $\widetilde{l}_1 = \frac{5}{49}$. It can be found that $\widetilde{\kappa} = \frac{1}{20} + \frac{3\times 5}{4\times 49} < 1$. Therefore, due to the fact that all the assumptions of Theorem 3.2 hold, problem (4.2) has a solution.

5 Conclusion

In this paper, a generalized boundary value problem for impulsive fractional differential equations involving Caputo fractional derivative in abstract space has been studied. A reasonable formula and definition of solutions for such problem is introduced. The new existence results are obtained based upon the techniques of measures of noncompactness and the fixed point theory.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors, FL and HW, contributed to each part of this study equally and read and approved the final version of the manuscript.

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